Balanced Incomplete Block Design.

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Definition of BIB Design

- An incomplete block design is said to be a balanced incomplete block (BIB) design if it satisfies the following conditions:
  i. The experimental material is divided into b blocks of k units each, different treatments being applied to the units in the same block.
  ii. There are t treatments each of which occurs in r blocks
  iii. Any two treatments occur together in exactly λ blocks
- The quantities t, b, r, k and λ are called the parameters of the BIBD design

Properties of BIB Design

- The following relations hold among the parameters, and even these are only necessary conditions for the existence of a BIB design:

$$rt = kb \tag{1}$$

$$\lambda(t-1) = r(k-1) \tag{2}$$

$$r > \lambda \tag{3}$$

$$b \geq t \tag{4}$$

- An incomplete block design (IBD) with the same number of treatments and blocks is called a symmetrical incomplete block design

Statistical Analysis of BIBD

- As usual, we assume that there are t treatments and b blocks. In addition, we assume that each block contains k treatments, that each treatment occurs r times in the design (or is replicated r times), and that there are $N = rt = bk$ total observations.

- Furthermore, the number of times each pair of treatments appears in the same block is

$$\lambda = \frac{r(k-1)}{t-1} \tag{5}$$

- The parameter λ has to be an integer.

Statistical Model

- The statistical model for the BIBD is:

$$y_{ij} = \mu + \tau + \beta_j + \epsilon_{ij} \tag{6}$$

where y_{ij} is the ith observation in the jth block, μ is the overall mean, τ_i is the effect of the ith treatment, β_j is the effect of the jth block and ϵ_{ij} is the NID$(0, \sigma^2)$ random error component.

- The total variability in the data can be expressed by the total corrected sum of squares

$$SS_T = \sum_i \sum_j y_{ij}^2 - \frac{y_{..}^2}{N} \tag{7}$$

- Total variability may be partitioned into:

$$SS_T = SS_{Treatment(adjusted)} + SS_{Blocks} + SS_E \tag{8}$$

Cont'd

- where the sum of squares for treatments is adjusted to separate the treatment and the block effects. This adjustment is necessary because each treatment is represented in a different set of r blocks.
- The block sum of squares is

$$SS_{Blocks} = \frac{1}{k}\sum_{j=1}^{b} y_{.j}^2 - \frac{y_{..}^2}{N} \qquad (9)$$

  where $y_{.j}$ is the total in the jth block. SS_{Blocks} has $b-1$ degrees of freedom.
- The adjusted treatment sum of squares is

$$SS_{Treatments(adjusted)} = \frac{k\sum_{i=1}^{a} Q_i^2}{\lambda t} \qquad (10)$$

Cont'd

- where Q_i is the adjusted total for the ith treatment which is computed as

$$Q_i = y_{i.} - \frac{1}{k}\sum_{j=1}^{b} n_{ij} y_{.j}, \quad i = 1, 2, ..., t \tag{11}$$

with $n_{ij} = 1$ if treatment i appears in block j and $n_{ij} = 0$ otherwise with $t - 1$ degrees of freedom

- The error sum of squares is computed by subtraction

$$SS_E = SS_T - SS_{Treatment} - SS_{Blocks} \tag{12}$$

and has $N - t - b + 1$ degrees of freedom

- The appropriate statistic for testing the equality of the treatment effects is

$$F_0 = \frac{MS_{Treatment}}{MS_E}$$

ANOVA Table for BIBD

Source	SS	df	MSS	F_0
Treatment(adjusted)	$\frac{k\sum_{i=1}^{a} Q_i^2}{\lambda t}$	t-1	$\frac{SS_{Treatment}}{t-1}$	$F_0 = \frac{MS_{Treatmer}}{MS_E}$
Blocks	$\frac{1}{k}\sum_{j=1}^{b} y_{.j}^2 - \frac{y_{..}^2}{N}$	b-1	$\frac{SS_{Blocks}}{b-1}$	
Error	Subtraction	N-t-b+1	$\frac{SS_E}{N-t-b+1}$	
Total	$\sum_i \sum_j y_{ij}^2 - \frac{y_{..}^2}{N}$	N-1		

Example 1

Consider the data in Table 1 for the catalyst experiment. This is a BIBD with $t = 4, b = 4, k = 3, r = 3, \lambda = 2$, and N$=$ 12.

Treatment(catalyst)	Block (Batch of raw material)			
	1	2	3	4
1	73	74		71
2		75	67	72
3	73	75	68	
4	75		72	75

Determine if the catalyst employed has a significant effect on the time of reaction.

Solution

Treatment(catalyst)	Block (Batch of raw material)				
	1	2	3	4	$y_{i.}$
1	73	74		71	218
2		75	67	72	214
3	73	75	68		216
4	75		72	75	222
$y_{.j}$	221	224	207	218	$870 = y_{..}$

Cont'd

- The total sum of squares

$$SS_T = \sum_i \sum_j y_{ij}^2 - \frac{y_{..}^2}{12} = 63156 - \frac{(870)^2}{12} = 81$$

- The block sum of squares is

$$SS_{Blocks} = \frac{1}{3}\sum_{j=1}^{4} y_{.j}^2 - \frac{y_{..}^2}{12} = \frac{1}{3}[(221)^2 + (207)^2 + (224)^2 + (218)^2 - \frac{(870)^2}{12}]$$

- To compute the treatment sum of squares adjusted for blocks, we first determine the adjusted treatment totals as:

$$Q_1 = 218 - \frac{1}{3}(221 + 224 + 218) = -9/3$$

Cont'd

- Continuation:

$$Q_2 = 214 - \frac{1}{3}(207 + 224 + 218) = -7/3$$

$$Q_3 = 216 - \frac{1}{3}(207 + 221 + 224) = -4/3$$

$$Q_3 = 222 - \frac{1}{3}(207 + 221 + 218) = 20/3$$

- The adjusted sum of squares for treatment is:

$$SS_{Treatment(adjusted)} = \frac{k\sum_{i=1}^{4} Q_i^2}{\lambda t} = \frac{3([-9/3]^2 + ... + [20/3]^2)}{2 \times 4} = 22.75$$

- The error sum of squares

$$SS_E = SS_T - SS_{Treatment} - SS_{Blocks} = 3.25$$

ANOVA Table

Source	SS	df	MSS	F_0	P-value
Treatments	22.75	3	7.58	11.66	0.0107
Blocks	55.00	3			
Error	3.25	5	0.65		
Total	81.00	11			

Because the P-value is small(less than 0.05), we conclude that the catalyst employed has a significant effect on the time of reaction.

Example 2

- Consider the data from a taste panel experiment reported by Moskowitz (1988).
- This experiment is a BIB with t=4 levels of the treatment factor or recipe, and block size k=2

Data

Recipe				
Panelist	A	B	C	D
1	5	5	-	-
2	7	-	6	-
3	5	-	-	4
4	-	6	7	-
5	-	6	-	4
6	-	-	8	6
7	6	7	-	-
8	5	-	8	-
9	4	-	-	5
10	-	7	7	-
11	-	6	-	5
12	-	-	7	4

Solution

```r
library(daewr)
mod1 <- aov( score ~ panelist + recipe, data = taste)
summary(mod1)
```

```
##             Df Sum Sq Mean Sq F value Pr(>F)
## panelist    11 19.333  1.7576   2.301 0.1106
## recipe       3  9.125  3.0417   3.982 0.0465 *
## Residuals    9  6.875  0.7639
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1
```

Example 3

The experiment involves a Martindale wear tester which is a machine used to test the wearing quality of types of cloth or other fabrics. It possesses the feature that four pieces of cloth may be processed simultaneously in one run of the machine. The responses is weight loss in tenths of a milligram suffered by the test fabric when it is rubbed against a standard grade of emery paper for 1000 revolutions of the machine. The wearing quality of specimens of seven different types of cloth (treatments) A, B, C, D, E, F, G are to be compared. Four treatments are randomized and mounted in the four Martindale wear tester specimen holders.

Data

	Cloth Type							
Block	A	B	C	D	E	F	G	$y_{.j}$
1		627		248		563	252	1690
2	344		233			442	226	1245
3			251	211	160		297	919
4	337	537			195		300	1369
5		520	278		199	595		1592
6	369			196	185	606		1356
7	396	602	240	273				1511
$y_{i.}$	1446	2286	1002	928	739	2206	1075	9682

Least Squares Estimation of the parameters

- Estimating the treatment effects of the BIBD model, the least squares normal equations are:

$$\mu : N\hat{\mu} + r\sum_{i=1}^{t}\hat{\tau}_i + k\sum_{j=1}^{k}\hat{\beta}_j = y_{..} \tag{13}$$

$$\tau_i : r\hat{\mu} + r\hat{\tau}_i + \sum_{j=1}^{b} n_{ij}\hat{\beta}_j = y_{i.} \quad i = 1, 2, ..., t \tag{14}$$

$$\beta_j : k\hat{\mu} + \sum_{i=1}^{t} n_{ij}\hat{\tau}_i + k\hat{\beta}_j = y_{.j} \quad j = 1, 2, ..., b \tag{15}$$

- Imposing $\sum\hat{\tau}_i = \sum\hat{\beta}_j = 0$ we find that $\hat{\mu} = \bar{y}_{...}$

Solving in R

- We can conduct the analysis of BIBD in R programming language.
- We need the following packages to be installed: `ibd`,`crossdes`, `multcomp`

Thank You!